

REAL BAUM-CONNES ASSEMBLY AND T-DUALITY FOR TORUS ORIENTIFOLDS

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ABSTRACT. We show that the real Baum-Connes conjecture for abelian groups, possibly twisted by a cocycle, explains the isomorphisms of (twisted) KR -groups that underlie all T-dualities of torus orientifold string theories.

1. INTRODUCTION

This paper was motivated by joint work with Charles Doran and Stefan Mendez-Diez [7, 8], in which we studied type II orientifold string theories on circles and 2-tori. In these theories, D-brane charges lie in twisted KR -groups of (X, ι) , where X is the spacetime manifold and ι is the involution on X defining the orientifold structure. (That D-brane charges for orientifolds are classified by KR -theory was pointed out in [32, §5.2], [13], and [10], but twisting (as defined in [19, 18, 20] and [7]) may arise due to the B -field, as in [33], and/or the charges of the O -planes, as explained in [7].) These orientifold theories were found in [8] to split up into a number of T-duality groupings, with the theories in each grouping all related to one another by various T-dualities. The twisted KR -groups attached to each of the theories within a T-duality grouping were all found to be isomorphic to one another, up to a degree shift.

One thing that was missing in this previous work was a mathematical explanation for these twisted KR isomorphisms. The purpose of this paper is to provide such an explanation. In fact, it turns out that the isomorphisms of KR -groups associated with the T-dualities for torus orientifolds come from the real Baum-Connes assembly maps for abelian groups, possibly twisted by a cocycle. Thus these T-dualities may be explained mathematically by the fact that the real Baum-Connes conjecture is valid for these cases [3, 30].

Since the appearance of the Baum-Connes conjecture in this context might seem surprising and unmotivated, we should perhaps explain why we were led to look at the Baum-Connes assembly map. There are two justifications. One is that in [17] and in [28], we found that the K -theory isomorphisms associated to T-duality can often

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explained via the calculation of K -theory of crossed products by \mathbb{R} using Connes' "Thom isomorphism" theorem of [4]. This theorem of Connes is now recognized as being a special case of a proof of the Baum-Connes conjecture with coefficients (in the case $G = \mathbb{R}$). While the cases treated in [17, 28] deal with complex K -theory, duality involving KR -theory appeared in [27, Theorem 2.5] in the case where the group involved is free abelian. In that theorem, real Baum-Connes was shown to give an isomorphism from KO -homology of a torus to KR -cohomology of another torus of the same dimension (really the T-dual torus). This isomorphism turns out (in Proposition 1 below) to be related to T-duality between the type I and type IA string theories on the circle.

2. THE MAIN CONSTRUCTION AND RESULTS

We will be working throughout with Atiyah's KR -theory [1]. This is the topological K -theory (with compact supports) of Real vector bundles E over Real spaces (X, ι) . A Real space is just a locally compact Hausdorff space X equipped with a self-homeomorphism ι satisfying $\iota^2 = \text{id}_X$. A Real vector bundle E over such a space is a complex vector bundle equipped with a conjugate-linear vector bundle automorphism of period 2 compatible with ι . The KR -theory of (X, ι) can be identified with the topological K -theory of the real Banach algebra $C_0(X, \iota) = \{f \in C_0(X) \mid f(\iota(x)) = \overline{f(x)}\}$. We shall use the indexing convention of [16, 7, 8]: $\mathbb{R}^{p,q}$ denotes $\mathbb{R}^p \oplus \mathbb{R}^q$ with the involution that is $+1$ on the first summand and -1 on the second summand, and $S^{p,q}$ denotes the unit sphere in $\mathbb{R}^{p,q}$.

2.1. Circle orientifolds. We begin with the case of orientifolds on a circle, with the involution coming from a linear involution on \mathbb{R}^2 , restricted to the unit circle. It was found in [9, 7, 8] that there are four such orientifold theories, known in the physics literature as types I, $\tilde{\text{I}}$, IA, and $\tilde{\text{IA}}$. These split into two T-duality groupings, one of which contains theories I and IA, corresponding to the Real spaces $S^{2,0}$ and $S^{1,1}$, and the other of which contains theories $\tilde{\text{I}}$ and $\tilde{\text{IA}}$, corresponding to the Real spaces $S^{0,2}$ and $S_{(+,-)}^{1,1}$. Here the subscript $(+, -)$ in $S_{(+,-)}^{1,1}$ indicates that of the two O -planes in $S^{1,1}$ (i.e., fixed points for the involution), one has been given a plus sign (meaning that the Chan-Paton bundle there is of real type) and one has been given a minus sign (meaning that the Chan-Paton bundle there is of quaternionic type).

Let us now see that the twisted KR isomorphisms in these two T-duality groupings amount to Baum-Connes assembly maps. The first group consists of theories I and IA, corresponding to the Real spaces $S^{2,0}$ and $S^{1,1}$. Thus we want an isomorphism $KR^{1-*}(S^{2,0}) \cong KR^{1-*}(S^{1,1})$. (The degree shift by 1 is explained by the fact that we are applying T-duality in a single circle, and thus going from a IIB theory to a IIA theory.) Now $S^{2,0}$ is simply S^1 with a trivial involution, which we can identify with the classifying space $B\mathbb{Z}$ of the infinite cyclic group \mathbb{Z} . The real Baum-Connes conjecture holds [3, 30] for amenable groups since the complex

Baum-Connes conjecture holds for these groups [11]. Thus Baum-Connes gives an assembly isomorphism

$$(1) \quad \mu: KO_j(B\mathbb{Z}) \xrightarrow{\cong} KO_j(C_{\mathbb{R}}^*(\mathbb{Z})).$$

We can “unpack” this as follows. $B\mathbb{Z} = S^1 = S^{2,0}$, so the left-hand side is $KO_j(S^1) \cong KO^{1-j}(S^1)$ by Poincaré duality, since S^1 is a spin manifold. On the other hand, for an abelian locally compact group G , the Fourier transform sends $L_{\mathbb{R}}^1(G)$ (as a $*$ -algebra with convolution multiplication) to a dense subalgebra of

$$C_0(\widehat{G}, \iota) = \left\{ f \in C_0(\widehat{G}) : f(x^{-1}) = \overline{f(x)} \text{ for } x \in \widehat{G} \right\},$$

where \widehat{G} is the Pontrjagin dual of G and ι is the involution given by group inversion. Taking C^* -completions gives $C_{\mathbb{R}}^*(G) \cong C_0(\widehat{G}, \iota)$. In the case of $G = \mathbb{Z}$, $\widehat{G} = \mathbb{T}$ and inversion ι on \mathbb{T} is complex conjugation, and thus

$$KO_j(C_{\mathbb{R}}^*(\mathbb{Z})) \cong KO_j(C(\mathbb{T}, \iota)) = KO_j(C(S^{1,1})) = KR^{-j}(S^{1,1}).$$

Putting everything together, we obtain

Proposition 1. *There is a natural isomorphism $KO^{1-j}(S^1) \rightarrow KR^{-j}(S^{1,1})$ given by the composite $\mu \circ \delta$, where μ is the real Baum-Connes assembly map of (1) for the discrete group \mathbb{Z} and where $\delta: KO^{1-j}(S^1) \rightarrow KO_j(S^1)$ is Poincaré duality (given analytically by Kasparov product with the class of the Dirac operator).*

Proof. All of this was outlined above. The statement and proof of analytical Poincaré duality may be found in [14, §4]. \square

The more interesting and subtle case comes from the other T-duality grouping, consisting of the theories of types $\widetilde{\text{I}}$ and $\widetilde{\text{IA}}$. We want an isomorphism $KR^{1-*}(S^{0,2}) \cong KR_{(+,-)}^{-*}(S^{1,1})$. (Here the $(+, -)$ decoration on the KR -groups was explained in [7, §4]. The groups $KR_{(+,-)}^{-*}(S^{1,1})$ are the topological K -theory of a real Banach algebra which is locally Morita equivalent to $C(S^{1,1})$ near the fixed point with the $+$ decoration and locally Morita equivalent to $C(S^{1,1}) \otimes \mathbb{H}$ near the fixed point with the $-$ decoration.) Such an isomorphism was obtained “experimentally” in [7, §4.1], but the treatment there didn’t really explain where this isomorphism comes from (except in terms of the physics interpretation using T-duality of orientifold theories).

Let $G = \langle a, b \mid ab = ba, b^2 = 1 \rangle$. This is an abelian group isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$. On this group we can define a (normalized) 2-cocycle ω with values in $O(1) = \{\pm 1\}$ by $\omega(a, b) = \omega(b, a) = \omega(b, b) = -1$, $\omega(a, a) = +1$. Let $C^*(G, \omega)$ be the (complex) group C^* -algebra of G twisted by this cocycle, and let $C_{\mathbb{R}}^*(G, \omega)$ be the corresponding real C^* -algebra.

Lemma 1. *With G and ω as just defined, $C^*(G, \omega)$ has spectrum S^1 and*

$$KO_j(C_{\mathbb{R}}^*(G, \omega)) \cong KR_{(+,-)}^{-j}(S^{1,1}).$$

Proof. We begin by noting that $C^*(G, \omega)$ is the universal C^* -algebra on two unitaries U and V satisfying $V^2 = -1$ and $UV = -VU$. An irreducible unitary ω -representation of G , when restricted to $H = \langle a \rangle$, must be a sum of two unitary characters $U \mapsto z$ and $U \mapsto -z$, $z \in \mathbb{T}$, since V conjugates U to $-U$. Then the usual ‘‘Mackey machine’’ argument shows that the representation is induced from one or the other of these characters, and so the spectrum of the C^* -algebra is naturally identified to the quotient, which is again a circle, of \mathbb{T} by the antipodal map $z \mapsto -z$.

Now the representations $U \mapsto z$, $z \neq 1$ or -1 , of H are not defined over \mathbb{R} and assemble in conjugate pairs to two-dimensional irreducible representations $U \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, $\theta = \arg z$, of H over \mathbb{R} . So we see that the Real involution on the spectrum of $C^*(G, \omega)$ must interchange z and \bar{z} , giving the Real space $S^{1,1}$. We need to determine what happens at the fixed points. At $z = \pm 1$ the associated representation of G is given by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and is defined over \mathbb{R} . Furthermore, these matrices generate $M_2(\mathbb{R})$, which is Morita equivalent to \mathbb{R} . So this point corresponds to an O^+ -plane.

When $z = \pm i$ (note that this is a fixed point for complex conjugation after we divide out by multiplication by -1), the associated representation of G is given by

$$U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$UV = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and the \mathbb{R} -span of I , U , V , and UV give a copy of the quaternions \mathbb{H} . So at this point we get an irreducible real representation of quaternionic type, and this point corresponds to an O^- -plane. Now we see that $C_{\mathbb{R}}^*(G, \omega)$ is an algebra satisfying the properties of [7, Theorem 1], whose topological K -theory gives $KR_{(+,-)}^{-j}(S^{1,1})$. \square

Theorem 1. *The real Baum-Connes isomorphism for the group G defined above, with twist by the cocycle ω , reduces to*

$$(2) \quad \mu: KSC_j \cong KO_j^{(G, \omega)}(\mathbb{R}) \xrightarrow{\cong} KO_j(C_{\mathbb{R}}^*(G, \omega)) \cong KR_{(+,-)}^{-j}(S^{1,1}).$$

Upon composition with Poincaré duality $KSC^{1-j} \rightarrow KSC_j$, this becomes the isomorphism of twisted KR groups of [7, §4.1] that underlies the T -duality between the orientifold theories of types $\tilde{\text{I}}$ and $\tilde{\text{IA}}$.

Proof. Note that the central extension of G by $O(1)$ classified by ω is $\tilde{G} = \langle \tilde{a}, \tilde{b} \mid \tilde{a}\tilde{b} = \tilde{b}^{-1}\tilde{a}, \tilde{b}^4 = 1 \rangle$. (Here \tilde{a} and \tilde{b} are lifts of a and b , respectively.) This is

a solvable group of the form $K \rtimes \mathbb{Z}$, where K is the torsion subgroup, a cyclic group of order 4 generated by \tilde{b} , and the generator \tilde{a} of \mathbb{Z} acts on K by conjugating \tilde{b} to $\tilde{b}^3 = \tilde{b}^{-1}$. So the left-hand and right-hand sides of the real Baum-Connes map for (G, ω) are direct summands in the corresponding sides of the real Baum-Connes map for \tilde{G} . Since \tilde{b}^2 is central of order 2, $C_{\mathbb{R}}^*(\tilde{G})$ splits as $C_{\mathbb{R}}^*(G, \omega) \oplus C_{\mathbb{R}}^*(G)$, where the two summands correspond to representations where \tilde{b}^2 takes the value -1 (resp., $+1$).

The left-hand side of the real Baum-Connes map for \tilde{G} is $KO_j^{\tilde{G}}(\mathbb{R})$, since \mathbb{R} , with the action where \tilde{a} acts by translation by 1 and \tilde{b} acts trivially, is the universal proper \tilde{G} -space. Furthermore, as pointed out by Kasparov [15, comments following Definition 5],

$$KO_j^{\tilde{G}}(\mathbb{R}) \cong KK O^{-j}(C_0^{\mathbb{R}}(\mathbb{R}) \rtimes \tilde{G}, \mathbb{R}),$$

so we need to understand the structure of the crossed product

$$C_0^{\mathbb{R}}(\mathbb{R}) \rtimes \tilde{G} \cong (C_0^{\mathbb{R}}(\mathbb{R}) \rtimes K) \rtimes \mathbb{Z}.$$

Since K acts trivially on \mathbb{R} ,

$$C_0^{\mathbb{R}}(\mathbb{R}) \rtimes K \cong C_0^{\mathbb{R}}(\mathbb{R}) \otimes C_{\mathbb{R}}^*(K) \cong C_0^{\mathbb{R}}(\mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}).$$

The two summands of \mathbb{R} in $C_{\mathbb{R}}^*(K)$ correspond to the representations $\tilde{b} \mapsto \pm 1$, which are trivial on \tilde{b}^2 . So these summands will go to $C_{\mathbb{R}}^*(G)$ on the right-hand side. The summand sent under μ to $C_{\mathbb{R}}^*(G, \omega)$ is thus:

$$(C_0^{\mathbb{R}}(\mathbb{R}) \otimes \mathbb{C}) \rtimes \mathbb{Z}.$$

But we have to be careful with the action of \mathbb{Z} . On $C_0^{\mathbb{R}}(\mathbb{R})$, the generator \tilde{a} of \mathbb{Z} acts by translation by 1. But on \mathbb{C} , which corresponds to the representations $\tilde{b} \mapsto \pm i$ of K , \tilde{a} acts by complex conjugation since conjugation by \tilde{a} sends \tilde{b} to \tilde{b}^{-1} . Thus for $f \in C_0(\mathbb{R})$, $\tilde{a} \cdot f(x) = \overline{f(x+1)}$. The crossed product $C_0(\mathbb{R}) \rtimes \mathbb{Z}$ is therefore Morita equivalent to

$$\{f \in C(\mathbb{R}/2\mathbb{Z}) \mid f(x+1) = \overline{f(x)}\} \cong C(S^{0,2}),$$

and μ reduces to an isomorphism

$$KR_j(S^{0,2}) \rightarrow KR_{(+,-)}^{-j}(S^{1,1}).$$

The left-hand side is isomorphic to $KR^{1-j}(S^{0,2}) = KSC^{1-j}$ via Poincaré duality for $S^{0,2}$, and the theorem follows via real Baum-Connes for solvable groups [3, 30] and Lemma 1. \square

2.2. 2-Torus orientifolds. In [9, 8], a study was made of all possible type II orientifold string theories on 2-tori. For reasons of supersymmetry, it was assumed that the 2-torus X on which spacetime was compactified is equipped with a complex structure making it into an elliptic curve, and that the orientifold involution ι on X is holomorphic in type IIB and antiholomorphic in type IIA. It turned out that there are ten distinct such theories, divided into 3 groupings. All the theories in a single grouping are related to one another by sequences of T-dualities. Some of these T-dualities had been predicted earlier, for example in [29, 33].

The first group contains the type I theory on T^2 (this is the IIB orientifold for the Real space (T^2, id_{T^2})), the IIA orientifold theory on T^2 for an orientation-reversing involution with fixed set $S^1 \amalg S^1$, and the IIB orientifold theory on T^2 for an orientation-preserving involution with 4-point fixed set, each point with positive O -plane charge. The associated Real spaces are $S^{2,0} \times S^{2,0}$, $S^{1,1} \times S^{2,0}$, and $S^{1,1} \times S^{1,1}$. The T-dualities within this group can all be easily obtained (by taking products) from the case of types I and IA on the circle (see Proposition 1), so we will not discuss them further.

The second group consists of the $\tilde{\text{I}}$ theory (the IIB orientifold on $S^{2,0} \times S^{0,2}$), the $\tilde{\text{IA}}$ theory on $S^{1,1} \times S^{2,0}$, the IIA theory on $S^{1,1} \times S^{0,2}$, and the IIB theory on T^2 for an orientation-preserving involution with 4-point fixed set, where half the fixed points have positive O -plane charge and half have negative O -plane charge. The T-dualities within this group can all be easily obtained (by taking products) from the case of types $\tilde{\text{I}}$ and $\tilde{\text{IA}}$ on the circle (see Theorem 1), along with the I-IA duality on the circle (Proposition 1), so again we will not discuss them further.

The interesting and subtle case involves the final T-duality grouping. There are three theories in this group, the type I theory with non-trivial B -field (called the theory “without vector structure” in [33]), the IIA theory for an antiholomorphic involution with fixed set S^1 and quotient space a Möbius strip, and the IIB orientifold theory for a holomorphic involution with four fixed points, three of which have positive O -plane charge and one of which has negative O -plane charge. In [7, 8], the twisted KR -groups for all of these theories were computed and found to be (abstractly) isomorphic, but no explicit isomorphisms of twisted KR -groups were obtained. We will now remedy this deficiency.

Let us now switch notation from Section 2.1 and let G be a free abelian group on two generators a and b , and let $\nu \in Z^2(G, O(1))$ be the normalized 2-cocycle on G with $\nu(a, b) = \nu(b, a) = -1$, $\nu(a, a) = \nu(b, b) = +1$. The associated central extension of G by $O(1)$ is $\tilde{G} = \langle \tilde{a}, \tilde{b}, c \mid \tilde{a}\tilde{b} = c\tilde{b}\tilde{a}, c^2 = 1 \rangle$. (Here \tilde{a} and \tilde{b} are lifts of a and b , respectively, and c , corresponding to $-1 \in O(1)$, is central.) The counterpart to Lemma 1 is the following:

Lemma 2. *With G and ν as just defined, $C^*(G, \nu)$ has spectrum T^2 and*

$$KO_j(C_{\mathbb{R}}^*(G, \nu)) \cong KR_{(+, +, +, -)}^{-j}(S^{1,1} \times S^{1,1}).$$

Proof. We begin by computing the complex C^* -algebra $C^*(G, \nu)$. This is the free C^* -algebra on two unitary generators U and V satisfying the commutation rule $UV = -VU$. In other words, it is just the (rational) noncommutative torus $A_{1/2}$. This has spectrum T^2 and is the algebra of sections of a stably trivial, locally trivial bundle of algebras over T^2 , with fibers isomorphic to $M_2(\mathbb{C})$ (see for instance [12, 5]). The center of $C^*(G, \nu)$ is generated by U^2 and V^2 , which together generate a copy of $C(T^2)$. If $z, w \in \mathbb{T}$, then in an irreducible representation with $U^2 \mapsto z$ and $V^2 \mapsto w$, the eigenvalues of U are $\pm z^{1/2}$ and the eigenvalues of V are $\pm w^{1/2}$. When either $z^{1/2}$ or $w^{1/2}$ is non-real, to get a real representation of $C_{\mathbb{R}}^*(G, \nu)$ we need to take the eigenvalues of U to be $\pm z^{1/2}, \pm \bar{z}^{1/2}$ and the eigenvalues of V to be $\pm w^{1/2}, \pm \bar{w}^{1/2}$. So from this analysis, we can see that the underlying Real space of $C_{\mathbb{R}}^*(G, \nu)$ must be $S^{1,1} \times S^{1,1}$. It remains to compute the O -plane charges at the fixed points, where $z = \pm 1$ and $w = \pm 1$. If $z = 1$ and $w = 1$, we have an irreducible representation given by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and this is defined over \mathbb{R} . Furthermore, these matrices generate $M_2(\mathbb{R})$, which is Morita equivalent to \mathbb{R} . So this point corresponds to an O^+ -plane. If $z = 1$ and $w = -1$ (or the other way around—these cases are symmetrical), then we have an irreducible representation given by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and this is defined over \mathbb{R} . Furthermore, these matrices generate $M_2(\mathbb{R})$, which is Morita equivalent to \mathbb{R} . So again these points correspond to O^+ -planes. But if $z = w = -1$, then just as in the proof of Lemma 1, we have an irreducible representation generated by

$$U = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These matrices generate a copy of the quaternions \mathbb{H} , and we have an O^- -plane. So $C_{\mathbb{R}}^*(G, \nu)$ is a real continuous-trace algebra with spectrum $S^{1,1} \times S^{1,1}$ and with three O^+ -planes, one O^- -plane. This algebra satisfies all the properties of [7, Theorem 1], and its topological K -theory gives $KR_{(+,+,+,-)}^{-j}(S^{1,1} \times S^{1,1})$. \square

Remark 1. The reader used to the theory of the Hilbert symbol will note that the sign choice $(+, +, +, -)$ arose here from the fact that if we compute the Hilbert symbol (z, w) for all choices $z, w \in \{\pm 1\}$, then it takes the value $+1$ when either z or w is positive and takes the value -1 exactly when $z = w = -1$.

Theorem 2. *The real Baum-Connes isomorphism for the group G defined above, with twist by the cocycle ν , reduces to*

(3)

$$\mu: KO_j(T^2, w) \cong KO_j^{(G, \nu)}(\mathbb{R}^2) \xrightarrow{\cong} KO_j(C_{\mathbb{R}}^*(G, \nu)) \cong KR_{(+, +, +, -)}^{-j}(S^{1,1} \times S^{1,1}).$$

Here w is the nontrivial element of $H^2(T^2, \mathbb{Z}/2)$ and $KO_*(T^2, w)$ denotes the associated twisted KO -homology. Upon composition with Poincaré duality

$$KO^{2-j}(T^2, w) \rightarrow KO_j(T^2, w),$$

this becomes the isomorphism of twisted KR groups of [7, §4.2 and §5] that underlies the T -duality between the “type I theory without vector structure” and the IIB orientifold theory on T^2 with four fixed points, three of them O^+ -planes and one of them an O^- -plane.

Proof. Once again, we use the fact that the left-hand side of the real Baum-Connes conjecture for (G, ν) is a direct summand in the left-hand side of the real Baum-Connes conjecture for \tilde{G} . The universal proper \tilde{G} -space is \mathbb{R}^2 , with \tilde{a} acting by translation by $(1, 0)$ and \tilde{b} acting by translation by $(0, 1)$. So, again following [15, comments following Definition 5],

$$KO_j^{\tilde{G}}(\mathbb{R}^2) \cong KK O^{-j}(C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes \tilde{G}, \mathbb{R}),$$

and we need to understand the structure of the crossed product

$$C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes \tilde{G} \cong (C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes G) \oplus (C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes_{\nu} G).$$

Here the first summand $C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes G$, where $c \in \tilde{G}$ acts by $+1$, corresponds to $C_{\mathbb{R}}^*(G)$ on the right-hand side of the Baum-Connes conjecture, and the other summand (the one we are interested in, where c acts by -1) corresponds to $C_{\mathbb{R}}^*(G, \nu)$. This summand $C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes_{\nu} G$ has complexification Morita equivalent to $C(T^2)$, since G acts freely on \mathbb{R}^2 with quotient T^2 , and since the complex Dixmier-Douady class has to vanish since $H^3(T^2, \mathbb{Z}) = 0$. So $C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes_{\nu} G$ is a real continuous-trace algebra with spectrum T^2 , with trivial involution.

We still need to compute the *real* Dixmier-Douady class. The calculation is the exact analogue of a result of Wassermann [31, Theorem 5] and Raeburn-Williams [25, Theorem 4.1] (see also [24, Remark on pp. 26–27]) in the complex case, except that we have to replace $U(1) = \mathbb{T}$ by $O(1)$. The result says that this class $w \in H^2(\mathbb{R}^2/\mathbb{Z}^2, \mathbb{Z}/2)$ is exactly the image of the class of ν in $H^2(BG, O(1))$, so it’s the nontrivial element of $H^2(T^2, \mathbb{Z}/2)$. Thus, via an application of Lemma 2, the real Baum-Connes isomorphism for (G, ν) reduces to

$$KO_j(T^2, w) \xrightarrow{\cong} KO_j(C_{\mathbb{R}}^*(G, \nu)) \cong KR_{(+, +, +, -)}^{-j}(S^{1,1} \times S^{1,1}).$$

Composing with Poincaré duality, we get the desired isomorphism

$$KO^{2-j}(T^2, w) \xrightarrow{\cong} KR_{(+, +, +, -)}^{-j}(S^{1,1} \times S^{1,1}).$$

The degree shift by 2 was explained in [8] by the fact that the associated type IIB orientifold string theories differ by a double T-duality. \square

2.3. The real Phillips-Raeburn obstruction. Before we get to the last case, we need a short digression on the real Phillips-Raeburn obstruction, which might be known to some people in the noncommutative geometry community but doesn't seem to be discussed in the literature. For simplicity, we will just deal with the case of automorphisms of $C_0(X, \mathcal{K}_{\mathbb{R}})$, where X is a second countable locally compact Hausdorff space, which in our case of interest will have the homotopy type of a finite CW complex, and $\mathcal{K}_{\mathbb{R}}$ denotes the compact operators on an infinite-dimensional separable real Hilbert space. Since every automorphism of $\mathcal{K}_{\mathbb{R}}$ is inner, $\text{Aut } \mathcal{K}_{\mathbb{R}} \cong PO = O/\{\pm 1\}$, the infinite-dimensional projective orthogonal group. Here O is given the strong (or weak) operator topology and is contractible, for the same reason as the infinite unitary group U [6, Lemme 3].

In the complex case, recall [26, §5.4] that $\text{Aut}_X C_0(X, \mathcal{K})$, the spectrum-fixing automorphisms, consists of *locally inner* automorphisms, and that the obstruction to a locally inner automorphism being inner is given by the Phillips-Raeburn obstruction in $H^2(X, \mathbb{Z})$ ([22], [26, Theorem 5.42]). This obstruction is easy to describe: a locally inner automorphism θ of $\text{Aut}_X C_0(X, \mathcal{K})$ can be viewed as a continuous map $\theta: X \rightarrow \text{Aut } \mathcal{K} \cong PU$, and since the projective unitary group is a classifying space for \mathbb{T} , and thus has the homotopy type of a $K(\mathbb{Z}, 2)$ space, the homotopy class $[\theta]$ of θ gives a class in $H^2(X, \mathbb{Z})$. Vanishing of this class is the obstruction to lifting the map θ to a map $X \rightarrow U$, and thus to θ being inner.

There is another way of understanding the Phillips-Raeburn obstruction [23], which is also relevant. An automorphism θ is the same thing as an action of \mathbb{Z} , and if θ is spectrum-fixing, $C_0(X, \mathcal{K}) \rtimes_{\theta} \mathbb{Z}$ is a continuous-trace algebra with spectrum Y which is a principal S^1 -bundle over X , the S^1 -action coming from the dual action of $\mathbb{T} = \widehat{\mathbb{Z}}$ on the crossed product. The class $[\theta] \in H^2(X, \mathbb{Z})$ is precisely the Chern class of this bundle.

Now let's consider the real case. The proof that every spectrum-fixing automorphism of $C_0(X, \mathcal{K}_{\mathbb{R}})$ is locally inner is exactly the same as in the complex case. A locally inner automorphism θ of $\text{Aut}_X C_0(X, \mathcal{K}_{\mathbb{R}})$ can be viewed as a continuous map $\theta: X \rightarrow \text{Aut } \mathcal{K}_{\mathbb{R}} \cong PO$, and since the projective orthogonal group is a classifying space for $O(1)$, and thus has the homotopy type of a $K(\mathbb{Z}/2, 1)$ space, the homotopy class $[\theta]$ of θ gives a class in $H^1(X, \mathbb{Z}/2)$. Vanishing of this class is the obstruction to lifting the map θ to a map $X \rightarrow O$, and thus to θ being inner.

As in the complex case, we can also describe the real Phillips-Raeburn obstruction as the class of a bundle obtained from the crossed product. Since the map $PO \rightarrow PU$ is null-homotopic, if we complexify, a locally inner automorphism θ becomes inner and $C_0(X, \mathcal{K}) \rtimes_{\theta_{\mathbb{C}}} \mathbb{Z}$ is isomorphic to $C_0(X, \mathcal{K}) \otimes C^*(\mathbb{Z}) \cong C_0(X \times S^1, \mathcal{K})$. In particular, the associated S^1 -bundle $X \times S^1 \rightarrow X$ is trivial. However, the triviality comes from the fact that we have forgotten the real structure. The spectrum

of $C_0(X, \mathcal{K}_{\mathbb{R}}) \rtimes_{\theta} \mathbb{Z}$ is a Real space, and over an open set $U \subseteq X$ where θ is inner, this Real space is $U \times S^{1,1}$, since $C_{\mathbb{R}}^*(\mathbb{Z}) \cong C(S^{1,1})$. Globally, $(C_0(X, \mathcal{K}_{\mathbb{R}}) \rtimes_{\theta} \mathbb{Z})^{\wedge}$ is a bundle $Y \rightarrow X$ of Real spaces over X , with fibers $S^{1,1}$, which is trivial after forgetting the Real structure. The fixed set for the involution on the total space of the bundle is thus a principal $\mathbb{Z}/2$ -bundle over X , i.e., a regular double cover, and the real Phillips-Raeburn obstruction is the characteristic class of this cover in $H^1(X, \mathbb{Z}/2)$. To see this, observe that if θ is inner, then the crossed product by θ is isomorphic to $C_0(X, \mathcal{K}_{\mathbb{R}}) \otimes C_{\mathbb{R}}^*(\mathbb{Z})$, which has spectrum $X \times S^{1,1}$ (as a Real space), and the bundle is trivial. In the other direction, suppose the $\mathbb{Z}/2$ -bundle over X is trivial. That means there is a global Real section of $Y \rightarrow X$, which we can identify with a continuous family of real representations of the crossed product. Since we already know that $\theta_{\mathbb{C}}$ is inner, θ is implemented by some strongly continuous $u: X \rightarrow U(\mathcal{H}_{\mathbb{R}} \otimes \mathbb{C})$ which is orthogonal modulo center, i.e., by some $u: X \rightarrow O \cdot \mathbb{T}$. Since we have a global family of real representations, we can take u to be pointwise orthogonal, and thus the real Phillips-Raeburn obstruction vanishes. So vanishing of the real Phillips-Raeburn obstruction in $H^1(X, \mathbb{Z}/2)$ is necessary and sufficient for vanishing of the $\mathbb{Z}/2$ -bundle over X , and so it must correspond precisely to the usual characteristic class of this bundle, which is the obstruction to triviality of the bundle of spectra in the category of Real spaces.

2.4. The species 1 IIA theory. There is one case left to handle, the one which in many respects is the most subtle. This is the T-duality between the type IIA orientifold theory associated to a species 1 real elliptic curve¹ and the type I theory twisted by a nontrivial B -field.

The species 1 IIA theory, from the point of view of KR -theory, corresponds to a Real space (T^2, ι) in the sense of Atiyah, where the underlying topological space is T^2 , and the involution ι is orientation-reversing, with fixed set S^1 and quotient space a Möbius strip. As explained in [8, §2 and §5.2.3], (T^2, ι) has a concrete realization as $(\mathbb{C}/\Lambda, z \mapsto \bar{z})$, where Λ is the lattice in \mathbb{C} generated by 1 and by $\tau = \frac{1}{2} + i\tau_2$, where $\tau_2 > 0$. What makes this case tricky is that $\text{Re } \tau = \frac{1}{2}$, so the lattice Λ is *not* rectangular. Note, however, that we have a sublattice $\Lambda' \subset \Lambda$ of index 2, where Λ' is generated by 1 and 2τ , or equivalently by 1 and $2i\tau_2$. So Λ' is rectangular. The involution $z \mapsto \bar{z}$ on \mathbb{C}/Λ' gives the Real space $S^{2,0} \times S^{1,1}$ studied above in Section 2.2. So (T^2, ι) has $S^{2,0} \times S^{1,1}$ as a double cover.

The desired T-duality will involve real Baum-Connes with coefficients for the group \mathbb{Z} . This conjecture is stated in [2, §9] (with the substitution of KO for KU), and is in fact a theorem in the case $G = \mathbb{Z}$, by [30], for example. First we need to explain what the coefficient \mathbb{Z} -algebra is.

¹For a smooth irreducible projective curve X defined over \mathbb{R} , the set of complex points is a connected compact Riemann surface and the set of real points is a topologically a finite disjoint union of circles. The *species* is the number of components of the set of real points.

Let $A = C(S^1, M_2(\mathbb{R}))$, which as a real C^* -algebra is obviously Morita equivalent to $C^\mathbb{R}(S^1)$, associated to the Real space $S^{2,0}$. We now equip this with an interesting action of \mathbb{Z} as follows. Identify $z = e^{2\pi it} \in \mathbb{T}$ with the rotation matrix

$$r(z) = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix} \in SO(2),$$

and define $\theta \in \text{Aut } A$ to be given by

$$(\theta f)(z) = r(z^{1/2})f(z)r(z^{1/2})^{-1}.$$

At first sight this seems ambiguous since $z \in \mathbb{T}$ has two square roots, $\pm e^{\pi it}$, but since they differ by $-1 \in Z(O(2))$, the inner automorphisms they define are the same and so θ is well defined, regardless of what choice one makes for the square root.

The interesting feature of the automorphism θ is that it is *locally inner* but not inner. From the discussion in Section 2.3 above, it should be clear that θ has nontrivial real Phillips-Raebrun class in $H^1(S^1, \mathbb{Z}/2)$.

Lemma 3. *With A and the automorphism θ as just defined, the crossed product $A \rtimes_\theta \mathbb{Z}$ is strongly Morita equivalent to the commutative real C^* -algebra associated to the Real space (T^2, ι) , where ι is the orientation-reversing involution associated to the species 1 IIA elliptic curve orientifold theory.*

Proof. We take the crossed product in stages. Since the Phillips-Raebrun obstruction of θ is 2-torsion, the Phillips-Raebrun obstruction of θ^2 vanishes, the subgroup $2\mathbb{Z} \subset \mathbb{Z}$ acts on A by inner automorphisms, and $A \rtimes 2\mathbb{Z}$ is Morita equivalent to $A \otimes_{\mathbb{R}} (2\mathbb{Z})$, a stably commutative real C^* -algebra corresponding to the Real space $S^{2,0} \times S^{1,1}$. The crossed product $A \rtimes_\theta \mathbb{Z}$ is then obtained (up to strong Morita equivalence) by taking a further crossed product by the quotient group $\mathbb{Z}/2\mathbb{Z}$. (Use the Packer-Raebrun trick of [21].) We will see shortly that taking this further crossed product amounts to dividing $S^{2,0} \times S^{1,1}$ by a free involution.

From the discussion in Section 2.3 above, the crossed product $A \rtimes_\theta \mathbb{Z}$ is Morita equivalent to an abelian real C^* -algebra, associated to a Real space (X, ι) . This space (X, ι) has as double cover the Real space associated to $A \rtimes_\theta 2\mathbb{Z}$, or $S^{2,0} \times S^{1,1}$. But by nontriviality of the Phillips-Raebrun invariant of θ , the natural projection map $(X, \iota) \rightarrow S^{2,0}$ does not have a Real section. However, we know that just as a bundle of topological spaces (i.e., forgetting the Real structure), X is a trivial S^1 -bundle over S^1 , and is thus a 2-torus. Furthermore, the inclusion $2\mathbb{Z} \subset \mathbb{Z}$ induces a map $C^*(2\mathbb{Z}) \rightarrow C^*(\mathbb{Z})$ or $C(S^1) \rightarrow C(S^1)$ which is dual to the covering map $z \mapsto z^2$ on \mathbb{T} . So the double covering $S^{2,0} \times S^{1,1} \rightarrow (X, \iota)$ is nontrivial on the $S^{1,1}$ factor. But it is also nontrivial on the $S^{2,0}$ factor, because of the nontriviality of the Phillips-Raebrun invariant. From our previous description of the covering map $\mathbb{C}/\Lambda' \rightarrow \mathbb{C}/\Lambda$, we recognize (X, ι) as being associated to the species 1 real elliptic curve. \square

Theorem 3. *The isomorphism $KO^{1-j}(T^2, w) \rightarrow KO^{-j}(X, \iota)$ associated to the T -duality between the “type I theory without vector structure” and the IIA orientifold associated to a species 1 real elliptic curve can be obtained as the composition*

$$\begin{aligned} KO^{1-j}(T^2, w) &\xrightarrow[\cong]{\text{Theorem 2}} KKO_{j+1}(C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes_{\nu} \mathbb{Z}^2, \mathbb{R}) \\ &\xrightarrow[\cong]{\cong} KKO_{j+1}((C_0(\mathbb{R}) \otimes A) \rtimes \mathbb{Z}, \mathbb{R}) \xrightarrow[\cong]{\text{Poinc. d.}} KKO_j^{\mathbb{Z}}(C_0(\mathbb{R}), A) \\ &\xrightarrow[\cong]{\text{Baum-Connes}} KO_j(A \rtimes_{\theta} \mathbb{Z}) \xrightarrow[\cong]{\text{Lemma 3}} KR^{-j}(T^2, \iota). \end{aligned}$$

Proof. The real Baum-Connes isomorphism for \mathbb{Z} with coefficients in (A, θ) sends $KKO_j^{\mathbb{Z}}(C_0(\mathbb{R}), A)$ to $KO_j(A \rtimes_{\theta} \mathbb{Z})$, which by Lemma 3 can be identified with $KR^{-j}(T^2, \iota)$. So let’s analyze the left-hand side, $KKO_j^{\mathbb{Z}}(C_0(\mathbb{R}), A)$. Since A is Morita equivalent to $C^{\mathbb{R}}(S^1)$ and the \mathbb{Z} -action θ is the identity on S^1 , this is isomorphic by Poincaré duality to $KKO_{j+1}^{\mathbb{Z}}(C_0(\mathbb{R}) \otimes A, \mathbb{R})$ ([15, Theorem 1] and [14, Theorem 4.10] — the degree shift by 1 comes from the fact that $\dim S^1 = 1$), which in turn is isomorphic to $KKO_{j+1}((C_0(\mathbb{R}) \otimes A) \rtimes \mathbb{Z}, \mathbb{R})$. Here \mathbb{Z} is acting on $C_0(\mathbb{R})$ by translations and on A by θ . On the other hand, we know by the proof of Theorem 2 that

$$\begin{aligned} (4) \quad KO^{1-j}(T^2, w) &\xrightarrow[\cong]{\text{Poincaré duality}} KO_{j+1}(T^2, w) \cong \\ &KKO_{j+1}(C_0^{\mathbb{R}}(\mathbb{R}^2) \rtimes_{\nu} \mathbb{Z}^2, \mathbb{R}) \cong KKO_{j+1}((C_0^{\mathbb{R}}(\mathbb{R}) \otimes A) \rtimes \mathbb{Z}, \mathbb{R}). \end{aligned}$$

The last isomorphism in (4) is obtained by decomposing the twisted crossed product by \mathbb{Z}^2 as a crossed product by \mathbb{Z} , which gives something Morita equivalent to $C_0^{\mathbb{R}}(\mathbb{R}) \otimes A$, followed by another crossed product by \mathbb{Z} . This second crossed product has \mathbb{Z} acting on \mathbb{R} by translations. The action on A is trivial on the spectrum $S^1 \cong \mathbb{R}/\mathbb{Z}$, but not inner because of the noncommutativity of \tilde{G} . So up to inner automorphisms it is given by θ , which represents the unique nontrivial Phillips-Raeburn class. Splicing all the isomorphisms together, the result follows. \square

REFERENCES

- [1] M. F. Atiyah. K -theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966.
- [2] Paul Baum, Alain Connes, and Nigel Higson. Classifying space for proper actions and K -theory of group C^* -algebras. In *C^* -algebras: 1943–1993 (San Antonio, TX, 1993)*, volume 167 of *Contemp. Math.*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.
- [3] Paul Baum and Max Karoubi. On the Baum-Connes conjecture in the real case. *Q. J. Math.*, 55(3):231–235, 2004.
- [4] A. Connes. An analogue of the Thom isomorphism for crossed products of a C^* -algebra by an action of \mathbb{R} . *Adv. in Math.*, 39(1):31–55, 1981.
- [5] Marc De Brabanter. The classification of rotational rotation C^* -algebras. *Arch. Math.*, 43:79–83, 1984.

- [6] Jacques Dixmier and Adrien Douady. Champs continus d'espaces hilbertiens et de C^* -algèbres. *Bull. Soc. Math. France*, 91:227–284, 1963. available at www.numdam.org.
- [7] Charles Doran, Stefan Mendez-Diez, and Jonathan Rosenberg. T-duality for orientifolds and twisted KR -theory. *Lett. Math. Phys.*, 104(11):1333–1364, 2014, arXiv:1306.1779.
- [8] Charles Doran, Stefan Mendez-Diez, and Jonathan Rosenberg. String theory on elliptic curve orientifolds and KR -theory. *Comm. Math. Phys.*, to appear, arXiv:1402.4885.
- [9] Dongfeng Gao and Kentaro Hori. On the structure of the Chan-Paton factors for D-branes in type II orientifolds. preprint, 2010, arXiv:1004.3972.
- [10] Sergei Gukov. K -theory, reality, and orientifolds. *Comm. Math. Phys.*, 210:621–639, 2000, arXiv:hep-th/9901042.
- [11] Nigel Higson and Gennadi Kasparov. E -theory and KK -theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1):23–74, 2001.
- [12] Raphael Høegh-Krohn and Tor Skjelbred. Classification of C^* -algebras admitting ergodic actions of the two-dimensional torus. *J. Reine Angew. Math.*, 328:1–8, 1981.
- [13] Kentaro Hori. D-branes, T duality, and index theory. *Adv. Theor. Math. Phys.*, 3:281–342, 1999, hep-th/9902102.
- [14] G. G. Kasparov. Equivariant KK -theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988.
- [15] G. G. Kasparov. K -theory, group C^* -algebras, and higher signatures (conspectus). In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 101–146. Cambridge Univ. Press, Cambridge, 1995.
- [16] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
- [17] Varghese Mathai and Jonathan Rosenberg. T -duality for torus bundles with H -fluxes via noncommutative topology. *Comm. Math. Phys.*, 253(3):705–721, 2005, arXiv:hep-th/0401168.
- [18] E.-K. M. Moutuou. Twistings of KR for Real groupoids. 2011, arXiv:1110.6836.
- [19] E.-K. M. Moutuou. *Twisted groupoid KR -theory*. PhD thesis, Université de Lorraine, 2012. available at <http://www.theses.fr/2012LORR0042>.
- [20] E.-K. M. Moutuou. Graded Brauer groups of a groupoid with involution. *J. Funct. Anal.*, 266(5):2689–2739, 2014, arXiv:1202.2057.
- [21] Judith A. Packer and Iain Raeburn. Twisted crossed products of C^* -algebras. *Math. Proc. Cambridge Philos. Soc.*, 106(2):293–311, 1989.
- [22] John Phillips and Iain Raeburn. Automorphisms of C^* -algebras and second Čech cohomology. *Indiana Univ. Math. J.*, 29(6):799–822, 1980.
- [23] John Phillips and Iain Raeburn. Crossed products by locally unitary automorphism groups and principal bundles. *J. Operator Theory*, 11(2):215–241, 1984.
- [24] Iain Raeburn and Jonathan Rosenberg. Crossed products of continuous-trace C^* -algebras by smooth actions. *Trans. Amer. Math. Soc.*, 305(1):1–45, 1988.
- [25] Iain Raeburn and Dana P. Williams. Pull-backs of C^* -algebras and crossed products by certain diagonal actions. *Trans. Amer. Math. Soc.*, 287(2):755–777, 1985.
- [26] Iain Raeburn and Dana P. Williams. *Morita equivalence and continuous-trace C^* -algebras*, volume 60 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- [27] Jonathan Rosenberg. C^* -algebras, positive scalar curvature, and the Novikov conjecture. III. *Topology*, 25(3):319–336, 1986.

- [28] Jonathan Rosenberg. *Topology, C^* -algebras, and string duality*, volume 111 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2009.
- [29] Augusto Sagnotti. Open strings and their symmetry groups. In *Nonperturbative quantum field theory (Cargèse, 1987)*, volume 185 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 521–528. Plenum, New York, 1988, arXiv:hep-th/0208020.
- [30] Thomas Schick. Real versus complex K -theory using Kasparov’s bivariant KK -theory. *Algebr. Geom. Topol.*, 4:333–346, 2004.
- [31] Antony John Wassermann. *Automorphic actions of compact groups on operator algebras*. PhD thesis, University of Pennsylvania, 1981.
- [32] Edward Witten. D-branes and K -theory. *J. High Energy Phys.*, 1998(12):019, 1998, arXiv:hep-th/9810188.
- [33] Edward Witten. Toroidal compactification without vector structure. *J. High Energy Phys.*, (2):Paper 6, 43 pp. (electronic), 1998, arXiv:hep-th/9712028.

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